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# **Anchoring Heuristic in Option Pricing**

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# Anchoring Heuristic in Option Pricing<sup>1</sup>

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An anchoring-adjusted option pricing model is developed in which the risk of the underlying stock is used as a starting point that gets adjusted upwards to estimate call option risk. Anchoring bias implies that such adjustments are insufficient. Black-Scholes formula is obtained with no anchoring bias. The new model provides a unified explanation for a number of option pricing puzzles including the implied volatility skew, superior historical performance of covered call writing, and worse-than-expected performance of zero beta straddles. The model is consistent with recent empirical findings regarding leverage adjusted option returns, and extends easily to jump-diffusion and stochastic-volatility approaches.

***JEL Classification:*** G13, G12, G02

***Keywords:*** Anchoring, Implied Volatility Skew, Stochastic Volatility, Jump Diffusion, Covered Call Writing, Zero-Beta Straddle, Leverage Adjusted Option Returns, Behavioral Finance

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## Anchoring Heuristic in Option Pricing

One of the major achievements in financial economics is the no-arbitrage approach of pricing options that does not require careful modeling of investor demand. Starting from the Black-Scholes model (1973), this literature has expanded in various directions and most notably, has explored the implications of allowing for both jump diffusion and stochastic volatility. Bates (2003) reviews this literature and concludes that it does not adequately capture or explain the key observed features of empirical option prices. Jackwerth (2000) presents the intriguing finding that risk aversion functions recovered from option prices are irreconcilable with a representative investor. Perhaps, as argued in Bollen and Whaley (2004), another line of inquiry is to acknowledge the importance of heterogeneous expectations and the impact of resulting demand pressures on option prices.

In demand-based option pricing, investor risk perceptions naturally matter for option prices. A relevant question is how are such risk judgments formed? According to asset pricing theory, an investor is a subjective expected utility maximizer who forms his risk judgment by figuring out the covariance of an asset's return with his marginal utility of consumption. The investor forms such judgments for all available assets in the process of creating a portfolio that maximizes his expected utility from consumption. It is reasonable to think that instead of forming such judgments in isolation for each asset, such a risk judgment is formed for a familiar asset and then extrapolated to another closely related asset. For example, a call option and its underlying stock are closely related assets. As a call option is equivalent to a leveraged position in the underlying stock, it is reasonable to start with the risk of the underlying stock and add to it to form a risk judgment about a call option.

However, using the risk of the underlying stock as a starting point to arrive at the risk of a call option exposes one to the anchoring bias. Starting from the early experiments in Kahneman and Tversky (1974) that show that *adjustments in assessments away from some initial value are often insufficient*, over 40 years of research has demonstrated the relevance of anchoring in a variety of decision contexts (see Furnham, A., and Boo, H. C. (2011) for a literature review). Hirshleifer (2001) considers anchoring to be an important part of “dynamic psychology-based asset pricing theory in its infancy.” (p. 1535). The role of anchoring bias has been found to be important in equity markets

in how analysts forecast firms' earnings (see Cen, L. Hillary F., and Wei, J. (2013)). Also, Campbell, S. D., and Sharpe, S. A., (2009) find that expert consensus forecasts of monthly economic releases are systematically biased toward the value of previous months releases. Johnson, J., Liu, S., and Schnytzer, A. (2009) show that investors in a particular financial market (horse-race betting) are prone to the anchoring bias. Overall, anchoring appears to be a highly relevant concept for financial markets.

Anchoring bias implies that using the underlying stock risk as a starting point for assessing call option risk leads to an under-estimation of risk. This paper explores the implications of such a bias for option pricing. It puts forward an option pricing model that incorporates anchoring in the formation of risk judgments. The new option pricing model converges to the Black-Scholes model in the absence of anchoring bias. The new model is capable of explaining a variety of option pricing puzzles: 1) an explanation for the implied volatility skew, 2) an explanation for the superior historical performance of covered call writing, 3) an explanation for the worse-than-expected historical performance of zero-beta-straddles, and 4) an explanation for recent empirical findings regarding leverage adjusted index option returns.

Furthermore, the model predicts that average put returns (for options held to expiry) should be more negative than what the Black Scholes model predicts. This is quite intriguing given the empirical findings that average put returns (for options held to expiry) are typically more negative than what the popular option pricing models suggest (see Chambers, Foy, Liebner, & Lu (2014) and references therein). Also, with anchoring, average call returns are significantly smaller than Black Scholes prediction. Hence, the model also provides a potential explanation for the empirical findings in Coval and Shumway (2001) that call returns are a lot smaller than what they should be given their systematic risk.

There is considerable field and experimental evidence regarding the relevance of the anchoring bias for call option pricing. Market professionals with decades of experience often argue that a call option is a surrogate for the underlying stock.<sup>2</sup> Such opinions are surely indicative of the

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<sup>2</sup> As illustrative examples, see the following:

<http://finance.yahoo.com/news/stock-replacement-strategy-reduce-risk-142949569.html>  
<http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772>,  
<http://www.investingblog.org/archives/194/deep-in-the-money-options/>  
<http://www.triplescreenmethod.com/TradersCorner/TC052705.asp>,  
<http://daytrading.about.com/od/stocks/a/OptionsInvest.htm>

importance of the underlying stock risk as a starting point for thinking about a call option risk, and point to insufficient adjustment to perceived risk, which creates room for the surrogacy argument. Furthermore, a series of laboratory experiments (see Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011) show that the hypothesis that a call option is priced by equating its expected return to the expected return from the underlying stock outperforms other pricing hypotheses. The results are consistent with the idea that risk of the underlying stock is used as a starting point with the anchoring heuristic ensuring that adjustments to the stock risk to arrive at call risk are insufficient. Hence, expected call returns do not deviate from expected underlying stock returns as much as they should.

The central prediction of asset pricing theory is:

$$E[R_i] = R_F - \frac{1}{E[u'(c_{t+1})]} Cov[u'(c_{t+1}), R_i] \quad (0.1)$$

Where  $R_i$  and  $R_F$  denote the (gross) return on a risky asset and the return on the risk free asset respectively. Equation (0.1) shows that the return that a subjective expected utility maximizer expects from a risky asset depends on his belief about the covariance of the asset's return with his marginal utility of consumption.

According to (0.1), an investor is required to form a judgment about the covariance of an asset's return with his marginal utility of consumption. It is reasonable to think that instead of forming such judgments in isolation for each asset, such a judgment is formed for a familiar asset and then extrapolated to another similar asset. A call option derives its existence from the underlying stock, and their payoffs are strongly related and move together.

With the above in mind, an analogy maker is defined as *a subjective expected utility maximizer who uses the risk of the underlying stock as a reference point for forming risk judgments about the corresponding call options*. That is, an analogy maker assesses call risk in comparison with the underlying stock risk. If one forms a judgment about call option risk in comparison with his judgment about the underlying stock risk, then one may write:

$$Cov\left(\frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_c\right) = Cov\left(\frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_s\right) + \epsilon \quad (0.2)$$

Where  $R_C$  and  $R_S$  are call and stock returns respectively, and  $\epsilon$  is the adjustment used to arrive at call option risk from the underlying stock risk. Almost always, assets pay more (less) when consumption is high (less), hence, the covariance between an asset's return and marginal utility of consumption is negative. That is, I assume that  $Cov(u'(c_{t+1}), R_S) < 0$ . So, in order to make a call option at least as risky as the underlying stock, I assume  $\epsilon \leq 0$ . An analogy maker understands that *a call option is a leveraged position in the underlying stock, hence is riskier. However, starting from the risk of the underlying stock, he does not fully adjust for the risk, so he underestimates the risk of a call option.* That is,  $|\epsilon|$ , which is the absolute value of risk adjustment is not large as it should be.

In contrast, option pricing theory predicts that:

$$Cov\left(\frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_C\right) = \pi \cdot Cov\left(\frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_S\right) \quad (0.3)$$

Where  $\pi > 1$  and typically takes very large values. That is, typically  $\pi \gg 1$ . To appreciate, the difference between (0.3) and (0.2), note that under the Black Scholes assumptions,  $\pi = \Omega$ , which is call price elasticity w.r.t the underlying stock price.  $\Omega$  takes very large values, especially for out-of-the-money call options. That is  $Cov(u'(c_{t+1}), R_C)$  is likely to be a far bigger negative number with correct risk judgment than with analogy making. Hence, a comparison of (0.2) and (0.3) indicates that, with analogy making, one likely remains anchored to the risk of the underlying stock while forming risk judgments about the call option leading to underestimation of its risk.

Substituting (0.2) in (0.1) leads to:

$$E[R_C] = E[R_S] + |\epsilon| \quad (0.4)$$

In contrast, substituting (0.3) in (0.1) under the Black Scholes assumptions yields:

$$E[R_C] = R_F + \Omega \cdot (E[R_S] - R_F) \quad (0.5)$$

$\Omega > 1$  and typically takes very large values. Hence, expected call return is likely to be smaller under analogy making when compared with the Black-Scholes predictions.

If analogy makers influence call prices, shouldn't a rational arbitrageur make money at their expense by taking an appropriate position in the call option and the corresponding replicating portfolio in accordance with the Black Scholes model? Such arbitraging is difficult if not impossible

in the presence of transaction costs. Barberis and Thaler (2002) argue that the absence of “free lunch” does not imply that prices are right due to various limits to arbitrage including transaction costs. In our context, the presence of transaction costs is likely to eliminate any “free lunch” at the expense of analogy makers. It is worth mentioning that bid-ask spreads in ATM index options are typically of the order of 3 to 5% of the option price, and spreads are typically 10% of the option price for deep OTM index options. In continuous time, no matter how small the transaction costs are, the total transaction cost of successful replication grows without bound rendering the Black-Scholes argument toothless. It is well known that there is no non-trivial portfolio that replicates a call option in the presence of transaction costs in continuous time (see Soner, Shreve, and Cvitanic (1995)). In discrete time, transaction costs are bounded, however, a no-arbitrage interval is created. If analogy price lies within the interval, analogy makers cannot be arbitrated away. We show the conditions under which this happens in a binomial setting.

If rational investors cannot sell options and buy replicating portfolios to make money due to transaction costs, can they at least write naked options, and make money, on average, in the long run at analogy makers’ expense? Analogy makers over-value call options (and put options via put-call parity), so rational investors should write them. In the case of an adverse price movement, an option buyer can simply choose not to exercise an option whereas option writing creates an obligation. In particular, writing options creates bankruptcy risk for option writers if such investors are credit constrained; however, buying options does not create bankruptcy risk. It may not be possible for credit constrained rational writers to make money even in the long-run in the presence of bankruptcy risks at the expense of analogy makers (see Shleifer and Vishny (1997) for this argument in a general asset pricing setting).

Of course, if we move away from the framework of geometric Brownian motion to more realistic frameworks such as jump diffusion and stochastic volatility, then a replicating portfolio does not even exist, eliminating the possibility of any “free-lunch” at the expense of analogy makers irrespective of transaction costs and other limits to arbitrage. This article also shows that analogy making is complementary to the approaches developed earlier such as stochastic volatility and jump diffusion models. Such models specify certain dynamics for the underlying stock. The idea of analogy making is not wedded to a particular set of assumptions regarding the price and volatility processes of the underlying stock. It can be applied to a wide variety of settings. In this article, first we use the setting of a geometric Brownian motion. Then, we integrate analogy making with jump

diffusion and stochastic volatility approaches. Combining analogy making and stochastic volatility leads to the skew even when there is zero correlation between the stock price and volatility processes, and combining analogy making with jump diffusion generates the skew without the need for asymmetric jumps.

Section 2 builds intuition by providing a numerical illustration of call option pricing with analogy making. Section 3 develops the idea in the context of a one period binomial model. Section 4 puts forward the analogy based option pricing formulas in continuous time. Section 5 shows that if analogy making determines option prices, and the Black-Scholes model is used to back-out implied volatility, the skew arises, which flattens as time to expiry increases. Section 6 shows that the analogy model is consistent with key empirical findings regarding returns from covered call writing and zero-beta straddles. Section 7 shows that the analogy model is consistent with empirical findings regarding leverage adjusted option returns. Section 8 puts forward an analogy based option pricing model when the underlying stock returns exhibit stochastic volatility. It integrates analogy making with the stochastic volatility model developed in Hull and White (1987). Section 9 integrates analogy making with the jump diffusion approach of Merton (1976). Section 10 concludes.

## 2. Analogy Making: A Numerical Illustration

Consider an investor in a two state-two asset complete market world with one time period marked by two points in time: 0 and 1. The two assets are a stock (S) and a risk-free zero coupon bond (B). The stock has a price of \$140 today (time 0). Tomorrow (time 1), the stock price could either go up to \$200 (the red state) or go down to \$94 (the blue state). Each state has a 50% chance of occurring. There is a riskless bond (zero coupon) that has a price of \$100 today. Its price stays at \$100 at time 1 implying a risk free rate of zero. Suppose a new asset “A” is introduced to him. The asset “A” pays \$100 in cash in the red state and nothing in the blue state. How much should the investor be willing to pay for this new asset?

Finance theory provides an answer by appealing to the principle of no-arbitrage: *assets with identical state-wise payoffs must have the same price*. Consider a portfolio consisting of a long position in 0.943396 of S and a short position in 0.886792 of B. In the red state, 0.943396 of S pays \$188.6792 and one has to pay \$88.6792 due to shorting of 0.886792 of B earlier resulting in a net payoff of \$100. In the blue state, 0.943396 of S pays \$88.6792 and one has to pay \$88.6792 on account of shorting 0.886792 of B previously resulting in a net payoff of 0. That is, payoffs from 0.943396S-



0.886792B are identical to payoffs from “A”. As the cost of 0.943396S-0.886792B is \$43.39623, it follows that the no-arbitrage price for “A” is \$43.39623. Note, that the expected return from asset “A” with the no-arbitrage price is 15.22%. Asset “A” is equivalent to a call option on “S” with a striking price of \$100. The expected return from “S” is 5%. From (0.5), one can infer that  $\Omega = 3.044$ . That is, the call option is just over 3 times as risky as the underlying stock.

An analogy maker uses *the risk of the underlying stock as a reference point for forming risk judgments about call options*. This leads to a call option being priced in accordance with (0.4). As the expected stock return is 5%,  $|\epsilon|$  has to be slightly greater than 10%, if an analogy maker is to arrive at the correct expected return. More likely, due to anchoring heuristic,  $|\epsilon|$  is likely to be a lot smaller. We consider three cases: 1)  $|\epsilon| = 0$ , 2)  $|\epsilon| = 2.5\%$ , and 3)  $|\epsilon| = 5\%$ .

With  $|\epsilon| = 0$ , the analogy price of the call option is \$47.6190. That creates a gap of \$4.22277 between the no-arbitrage price and the analogy price. Rational investors should short “A” and buy “0.943396S-0.886792B”. However, transaction costs are ignored in the example so far.

Let’s see what happens when a symmetric proportional transaction cost of only 1% of the price is applied when assets are traded. That is, both a buyer and a seller pay a transaction cost of 1% of the price of the asset traded. Unsurprisingly, the composition of the replicating portfolio changes. To successfully replicate a long call option that pays \$100 in cash in the red state and 0 in the blue state with transaction cost of 1%, one needs to buy 0.952925 of S and short 0.878012 of B. In the red state, 0.952925S yields \$188.6792 net of transaction cost ( $200 \times 0.952925 \times (1 - 0.01)$ ), and one has to pay \$88.6792 to cover the short position in B created earlier ( $0.878012 \times 100 \times (1 + 0.01)$ ). Hence, the net cash generated by liquidating the replicating portfolio at time 1 is \$100 in the red state. In the blue state, the net cash from liquidating the replicating portfolio is 0. Hence, with a symmetric and proportional transaction cost of 1%, the replicating portfolio is “0.952925S-0.878012B”. The cost of setting up this replicating portfolio inclusive of transaction costs at time 0 is \$47.82044, which is larger than the price the analogy makers are willing to pay: \$47.6190. Hence, arbitrage profits cannot be made at the expense of analogy makers by writing a call and buying the replicating portfolio. The given scheme cannot generate arbitrage profits unless the call price is greater than \$47.82044

Suppose one is interested in doing the opposite. That is, buy a call and short the replicating portfolio to fund the purchase. Continuing with the same example, the relevant replicating portfolio (that generates an outflow of \$100 in the red state and 0 in the blue state) is “-0.934056S

+0.89575B". The replicating portfolio generates \$41.1928 at time 0, which leaves \$38.98937 after time 0 transaction costs in setting up the portfolio are paid. Hence, in order for the scheme to make money, one needs to buy a call option at a price less than \$38.98937.

Effectively, transaction costs create a no-arbitrage interval (38.98937, 47.82044). As the analogy price lies within this interval, arbitrage profits cannot be made at the expense of analogy makers in the example considered. With  $|\epsilon| = 2.5\%$ , the analogy price is \$46.51, and with  $|\epsilon| = 5\%$ , the analogy price is \$45.45. Clearly, such analogy makers cannot be arbitrated away even at transaction costs much smaller than 1%.

### ***2.1 Analogy Making: A Two Period Binomial Example with Delta Hedging***

Consider a two period binomial model. The parameters are: Up factor=2, Down factor=0.5, Current stock price=\$100, Risk free interest rate per binomial period=0, Strike price=\$30, and the probability of up movement=0.5. It follows that the expected gross return from the stock per binomial period is 1.25 ( $0.5 \times 2 + 0.5 \times 0.5$ ).

The call option can be priced both via analogy as well as via no-arbitrage argument. The no-arbitrage price is denoted by  $C_R$  whereas the analogy price is denoted by  $C_A$ . Define  $x_R = \frac{\Delta C_R}{\Delta S}$  and  $x_A = \frac{\Delta C_A}{\Delta S}$  where the differences are taken between the possible next period values that can be reached from a given node. For simplicity,  $|\epsilon| = 0$  is assumed. One can easily re-do the calculations for higher values of  $|\epsilon|$ .

Figure 1 shows the binomial tree and the corresponding no-arbitrage and analogy prices. Two things should be noted. Firstly, in the binomial case considered, before expiry, the analogy price is always larger than the no-arbitrage price. Secondly, the delta hedging portfolios in the two cases  $Sx_R - C_R$  and  $Sx_A - C_A$  grow at different rates. The portfolio  $Sx_A - C_A$  grows at the rate equal to the expected return on stock per binomial period (which is 1.25 in this case). In the analogy case, the value of delta-hedging portfolio when the stock price is 100 is 17.06667 ( $100 \times 0.98667 - 81.6$ ). In the next period, if the stock price goes up to 200, the value becomes 21.33333 ( $200 \times 0.98667 - 176$ ). If the stock price goes down to 50, the value also ends up being equal to 21.33333 ( $50 \times 0.98667 - 28$ ). That is, either way, the rate of growth is the same and is equal to

1.25 as  $17.06667 \times 1.25 = 21.33333$ . Similarly, if the delta hedging portfolio is constructed at any other node, the next period return remains equal to the expected return from stock. It is easy to verify that the portfolio  $Sx_R - C_R$  grows at a different rate which is equal to the risk free rate per binomial period (which is 0 in this case).

Put option values can be calculated easily via put-call parity. It is easy to check that the expected return from a put option (held to expiry) in this case has a value of -89.224% under analogy making. The corresponding value under portfolio replication is -43.75%. In general, under analogy making, expected return from a put option (held to expiry) is significantly more negative than what is expected from the Black Scholes model.

In the next section, implications of analogy making are explored in a binomial setting, which helps in building intuition before discussing the continuous time case.

Exp. Ret	1.25			Stock Price	400
Up Prob.	0.5				
Up	2				
Down	0.5				
Risk-Free r	0			$Call_R$	370
Strike	30			$Call_A$	370
		Stock Price	200		
		$x_R$	1		
		B	-30		
		$Call_R$	170		
		$x_A$	1		
		$Call_A$	176		
Stock Price	100			Stock Price	100
$x_R$	0.977778				
B	-25.5556				
$Call_R$	72.22222			$Call_R$	70
$x_A$	0.986667			$Call_A$	70
$Call_A$	81.6				
		Stock Price	50		
		$x_R$	0.933333		
		B	-23.3333		
		$Call_R$	23.33333		
		$x_A$	0.933333		
		$Call_A$	28		
				Stock Price	25
				$Call_R$	0
				$Call_A$	0

Figure 1

### 3. Analogy Making: The Binomial Case

Consider a two state world. The equally likely states are Red, and Blue. There is a stock with prices  $X_1$ , and  $X_2$  corresponding to states Red, and Blue respectively, where  $X_1 > X_2$ . The state realization takes place at time  $T$ . The current time is time  $t$ . We denote the risk free discount rate by  $r$ . That is, there is a riskless zero coupon bond that has a price of  $B$  in both states with a price of  $\frac{B}{1+r}$  today. For simplicity and without loss of generality, we assume that  $r = 0$  and  $T - t = 1$ . The current price of the stock is  $S$  such that  $X_1 > S > X_2$ . We further assume that  $S < \frac{X_1 + X_2}{2}$ . That is, the stock price reflects a positive risk premium. In other words,  $S = f \cdot \frac{X_1 + X_2}{2}$  where  $f = \frac{1}{1+r+\delta}$ .<sup>3</sup>  $\delta$  is the risk premium reflected in the price of the stock.<sup>4</sup> As we have assumed  $r = 0$ , it follows that  $f = \frac{1}{1+\delta}$ .

Suppose a new asset which is a European call option on the stock is introduced. By definition, the payoffs from the call option in the two states are:

$$C_1 = \max\{(X_1 - K), 0\}, C_2 = \max\{(X_2 - K), 0\} \quad (3.1)$$

Where  $K$  is the striking price, and  $C_1$ , and  $C_2$ , are the payoffs from the call option corresponding to Red, and Blue states respectively.

How much is an analogy maker willing to pay for this call option?

There are two cases in which the call option has a non-trivial price: 1)  $X_1 > X_2 > K$  and 2)  $X_1 > K > X_2$

The analogy maker infers the price of the call option,  $P_c$ , from (0.4):

$$\frac{\{C_1 - P_c\} + \{C_2 - P_c\}}{2 \times P_c} = \frac{\{X_1 - S\} + \{X_2 - S\}}{2 \times S} + \epsilon \quad (3.2)$$

As discussed in the introduction,  $\epsilon \leq 0$ . That is, either  $|\epsilon| = 0$  or  $|\epsilon| > 0$ .

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<sup>3</sup> In general, a stock price can be expressed as a product of a discount factor and the expected payoff if it follows a binomial process in discrete time (as assumed here), or if it follows a geometric Brownian motion in continuous time.

<sup>4</sup> If the marginal call investor is more optimistic than the marginal stock investor, they would perceive different values of  $X_1$  and  $X_2$  so that their assessment of  $\delta$  is different accordingly.

For case 1 ( $X_1 > X_2 > K$ ), if  $|\epsilon| = 0$ , one can write:

$$P_c = \frac{C_1 + C_2}{X_1 + X_2} \times S$$

$$\Rightarrow P_c = \left(1 - \frac{2K}{X_1 + X_2}\right) S \quad (3.3)$$

Substituting  $S = f \cdot \frac{X_1 + X_2}{2}$  in (3.3):

$$P_c = S - Kf \quad \text{if } |\epsilon| = 0 \quad (3.4a)$$

The above equation is the one period analogy option pricing formula for the binomial case when call remains in-the-money in both states, and  $|\epsilon| = 0$ .

Similarly,

$$P_c = \frac{1}{(1 + |\epsilon|f)} (S - Kf) \quad \text{if } |\epsilon| > 0 \quad (3.4b)$$

The corresponding no-arbitrage price  $P_r$  is (from the principle of no-arbitrage):

$$P_r = S - K \quad (3.5)$$

For case 2 ( $X_1 > K > X_2$ ), the analogy price is:

$$P_c = S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \quad \text{if } |\epsilon| = 0 \quad (3.6a)$$

And,

$$P_c = \frac{1}{(1 + |\epsilon|f)} \left( S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \right) \quad \text{if } |\epsilon| > 0 \quad (3.6b)$$

The corresponding no-arbitrage price is:

$$P_r = \frac{X_1 - K}{X_1 - X_2} (S - X_2) \quad (3.7)$$

**Proposition 1** *If  $|\epsilon| = 0$ , the analogy price is larger than the corresponding no-arbitrage price if the analogy maker expects a positive risk premium from the underlying stock. If  $|\epsilon| > 0$ , there is a threshold value  $|\bar{\epsilon}|$  below which the analogy price stays larger than the no-arbitrage price.*

**Proof.**

See Appendix A

■

Suppose there are transaction costs, denoted by “ $c$ ”, which are assumed to be symmetric and proportional. That is, if the stock price is  $S$ , a buyer pays  $S(1 + c)$  and a seller receives  $S(1 - c)$ . Similar rule applies when the bond or the option is traded. That is, if the bond price is  $B$ , a buyer pays  $B(1 + c)$  and a seller receives  $B(1 - c)$ . We further assume that the call option is cash settled. That is, there is no physical delivery.

The analogy price with transaction costs is given below:

$$P_c = \frac{1}{(1 + |\epsilon|(1 + c)f)} (S - Kf) \quad \text{if } X_1 > X_2 > K$$

$$P_c = \frac{1}{(1 + |\epsilon|(1 + c)f)} \left( S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \right) \quad \text{if } X_1 > K > X_2$$

The cost of replicating a call option changes when transaction costs are allowed. The total cost of successfully replicating a long position in the call option by buying the appropriate replicating portfolio and then liquidating it in the next period to get cash (as call is cash settled) is:

$$\left( \frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1 - c} - \frac{X_2}{1 + c} \right\} + c \left\{ \frac{S}{1 - c} + \frac{X_2}{1 + c} \right\} \quad \text{if } X_1 > K > X_2 \quad (3.8)$$

$$\left\{ \frac{S}{1 - c} - \frac{K}{1 + c} \right\} + c \left\{ \frac{S}{1 - c} + \frac{K}{1 + c} \right\} \quad \text{if } X_1 > X_2 > K \quad (3.9)$$

The corresponding inflow from shorting the appropriate replicating portfolio to fund the purchase of a call option is:

$$\left(\frac{X_1 - K}{X_1 - X_2}\right) \left\{ \frac{S}{1+c} - \frac{X_2}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{X_2}{1-c} \right\} \quad \text{if } X_1 > K > X_2 \quad (3.10)$$

$$\left\{ \frac{S}{1+c} - \frac{K}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{K}{1-c} \right\} \quad \text{if } X_1 > X_2 > K \quad (3.11)$$

Proposition 2 shows that if transaction costs exist and the risk premium on the underlying stock is within a certain range, the analogy price lies within the no-arbitrage interval. Hence, riskless profit cannot be earned at the expense of analogy makers.

**Proposition 2** *If  $|\epsilon| = 0$ , in the presence of symmetric and proportional transaction costs, analogy makers cannot be arbitrated out of the market if the risk premium on the underlying stock satisfies:*

$$0 \leq \delta \leq \frac{(1-c)(1+c)}{(1-c)^2 - 2\frac{S}{K}c(1+c)} - 1 \quad \text{if } X_1 > X_2 > K \quad (3.12)$$

$$0 \leq \delta \leq$$

$$\frac{\frac{K(X_1^2 - X_2^2)(1-c^2)}{2}}{X_2(X_1 - K)(X_1 + X_2)(1-c)^2 - S\{(1+c)^2(X_1^2 - X_2^2) - X_1(X_1 - X_2)(1-c^2)\}} - 1$$

*if  $X_1 > K > X_2$*  (3.13)

*If  $|\epsilon| > 0$ , the right hand limits in (3.12) and (3.13) are even larger.*

**Proof.**

See Appendix B

■



When transaction costs are introduced, there is no unique no-arbitrage price. Instead, a whole interval of no-arbitrage prices comes into existence. Proposition 2 shows that the analogy price may be within this no-arbitrage interval in a one period binomial setting. As more binomial periods are added, the transaction costs increase further due to the need for additional re-balancing of the replicating portfolio. In the continuous limit, the total transaction cost is unbounded. Reasonably, arbitrageurs cannot make money at the expense of analogy makers in the presence of transaction costs ensuring that the analogy makers survive in the market.

It is interesting to consider the rate at which the delta-hedged portfolio grows under analogy making. Proposition 3 shows that under analogy making, the delta-hedged portfolio grows at a rate  $\frac{1}{f} - 1 = r + \delta$  if  $|\epsilon| = 0$ , with the growth rate falling as  $|\epsilon|$  rises. This is in contrast with the Black Scholes Merton/Binomial Model in which the growth rate is equal to the risk free rate,  $r$ .

**Proposition 3** *If analogy making determines the price of the call option, then the corresponding delta-hedged portfolio grows with time at the rate of  $\frac{1}{f} - 1$  if  $|\epsilon| = 0$ , with the growth rate falling as  $|\epsilon|$  rises. In particular, the growth rate is equal to the risk free rate if  $|\epsilon| = (\Omega - 1)\delta$ , where  $\Omega$  is call price elasticity w.r.t the underlying stock price.*

**Proof.**

See Appendix C

■

The continuous limit of the discrete case discussed here leads to an option pricing formula, which is presented in the next section.

#### 4. Analogy Making: The Continuous Case

We maintain all the assumptions of the Black-Scholes model except two. Firstly, we assume that the risk of the underlying stock is used as a reference point for forming judgment about the risk of a call

option defined on the stock (analogy making). Secondly, we allow for proportional and symmetric transaction costs. As discussed in the introduction, analogy making leads to expected call option return given in (0.4). For a call option with strike  $K$ , over a small time interval,  $dt$ , one may write (with transaction costs):

$$\frac{E[dC]}{(1+\theta)C} = \frac{E[dS]}{(1+\theta)S} + |\epsilon_K| \quad (4.1)$$

Where  $C$ ,  $S$ , and  $\theta$  denote call price, stock price, and percentage transaction cost respectively.

If the risk free rate is  $r$  and the risk premium on the underlying stock is  $\delta$ , then,  $\frac{E[dS]}{S} = \mu = r + \delta$ .

To a given investor, a higher strike call is riskier than the lower strike call, so  $|\epsilon_K|$  may rise with strike.

Hence, (4.1) may be written as:

$$\frac{E[dC]}{C} = (r + \delta + |\epsilon_K|(1 + \theta)) \quad (4.2)$$

The underlying stock price follows geometric Brownian motion:

$$dS = \mu S dt + \sigma S dZ$$

Where  $dZ$  is the standard Guass-Weiner process.

From Ito's lemma:

$$E[dC] = \left( \mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt \quad (4.3)$$

Substituting (4.3) in (4.2) leads to:

$$(r + \delta + |\epsilon_K|(1 + \theta))C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r + \delta)S + \frac{\partial^2 C}{\partial S^2} \frac{\sigma^2 S^2}{2} \quad (4.4)$$

(4.4) describes the partial differential equation (PDE) that must be satisfied if analogy making determines call option prices.

To appreciate the difference between analogy making PDE and the Black-Scholes PDE, consider the expected return under the Black-Scholes approach, which is given in (0.5). Over a small time interval,  $dt$ , one may re-write (0.5) as:

$$\frac{E[dC]}{C} = r + \Omega \cdot (\mu - r) \quad (4.5)$$

Substituting (4.3) in (4.5) and realizing that  $\Omega = \frac{S}{C} \frac{\partial C}{\partial S}$  leads to the following:

$$rC = \frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{\partial^2 C}{\partial S^2} \frac{\sigma^2 S^2}{2} \quad (4.6)$$

(4.6) is the Black-Scholes PDE.

Comparing (4.4) and (4.6), it becomes immediately obvious that in a risk neutral world (with no transaction costs), the two PDEs are equal. After all, if expected returns from all assets are equal to the risk free rate then both  $\delta$  and  $\epsilon_K$  are zero. Risk aversion creates a wedge between the analogy price and the Black-Scholes price as with analogy making, the risk of a call option is under-estimated.

There is a closed form solution to the analogy PDE. Proposition 4 puts forward the resulting European option pricing formulas.

**Proposition 4** *The formula for the price of a European call is obtained by solving the analogy based PDE. The formula is  $C = e^{-|\epsilon_K|(1+\theta)(T-t)} \{SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)\}$*

*where  $d_1 = \frac{\ln(S/K) + (r+\delta+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$  and  $d_2 = \frac{\ln(\frac{S}{K}) + (r+\delta-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$*

**Proof.**

See Appendix D.

■

**Corollary 4.1** *The formula for the analogy based price of a European put option is*

$$Ke^{-r(T-t)} \{1 - e^{-\delta(T-t)}N(d_2)e^{-|\epsilon_K|(1+\theta)(T-t)}\} - S \left(1 - e^{-|\epsilon_K|(1+\theta)(T-t)}N(d_1)\right)$$

**Proof.**

Follows from put-call parity.

■

As proposition 4 shows, the analogy formula differs from the corresponding Black-Scholes formulas due to the appearance of  $\delta$ ,  $\epsilon_K$ , and  $\theta$ .

It is interesting to analyze put option returns under analogy making. Proposition 5 shows that put option returns are more negative under analogy making when compared with put option returns in the Black Scholes model.

**Proposition 5** *Expected put option returns (for options held to expiry) under analogy making are more negative than expected put option returns in the Black Scholes model as long as the underlying stock has a positive risk premium provided  $|\epsilon_K|(1 + \theta)$  does not cross a certain threshold*

**Proof.**

See Appendix I

■

Proposition 5 is quite intriguing given the puzzling nature of empirical put option returns when compared with the predictions of popular option pricing models. Chambers et al (2014) analyze nearly 25 years of put option data and conclude that average put returns are, in general, significantly more negative than the predictions of Black Scholes, Heston stochastic volatility, and Bates SVJ model. Hence, analogy making offers a potential explanation.

Of course, expected call return under analogy making is a lot less than what is expected under the Black Scholes model due to the anchoring bias. Empirical call returns are found to be a lot smaller given the predictions of the Black Scholes model (see Coval and Shumway (2001)). Hence,

analogy making is qualitatively consistent with the empirical findings regarding both call and put option returns.

## 5. The Implied Volatility Skew

If analogy making determines option prices (formulas in proposition 4), and the Black Scholes model is used to infer implied volatility, the skew is observed. For illustrative purposes, the following parameter values are used:  $S = 100$ ,  $T - t = 1 \text{ year}$ ,  $\sigma = 20\%$ ,  $r = 2\%$ ,  $\delta = 3\%$ , *and*  $\theta = 0$ . Note, the transaction cost  $\theta$  is assumed to be equal to zero to allow for the existence of the Black-Scholes price for comparison purposes.

An analogy maker uses the risk of the underlying stock as a starting point that gets adjusted upwards to arrive at the risk of a call option. Anchoring bias implies that the adjustment is not sufficient. Hence, the risk of a call option is under-estimated. Analogy makers understand that a call option is a leveraged position in the underlying stock which must get riskier as the striking price is increased. That is, the adjustment magnitude increases with strike price, however, it remains below the correct value.

With the assumed parameter values, it is possible to calculate the expected return from a call option under the Black-Scholes assumptions. As long as the adjustment made by analogy makers is smaller than the correct adjustment, the implied volatility skew is observed. If the adjustment is correct, that is, in accordance with the Black-Scholes model, then it must be:  $|\epsilon_K| = (\Omega_K - 1)\delta$ , where  $\Omega$  is call price elasticity w.r.t the underlying stock price. For the purpose of this illustration, assume that the actual adjustment is only a quarter of the correct adjustment. That is,  $|\epsilon_K| = 0.25(\Omega_K - 1)\delta$ . The resulting  $|\epsilon_K|$  values are shown in Table 1.

The expected returns under the Black Scholes assumptions as well as the resulting expected returns under analogy making, and corresponding implied volatilities are also shown in Table 1. Clearly, the implied volatility skew is observed.

Table 1						
K	Exp. Return (Black-Scholes)	Exp. Return (Analogy Making)	$ \epsilon_K $	Analogy Price	Black-Scholes Price	Implied Vol.
90	17.53%	8.13%	3.13%	16.18	14.93	24.35%
95	19.46%	8.62%	3.62%	12.87	11.76	23.47%
100	21.49%	9.12%	4.12%	10.03	9.08	22.85%
105	23.58%	9.65%	4.65%	7.66	6.87	22.39%
110	25.73%	10.18%	5.18%	5.74	5.10	22.05%

Figure 2 is a graphical illustration of the implied volatility skew seen in Table 1.

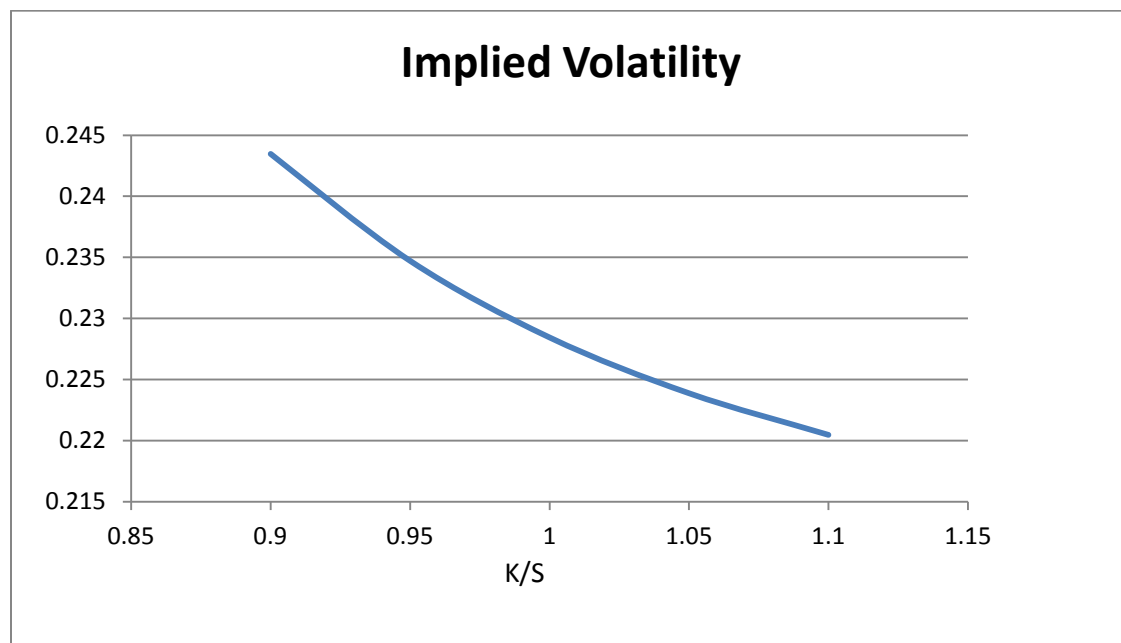
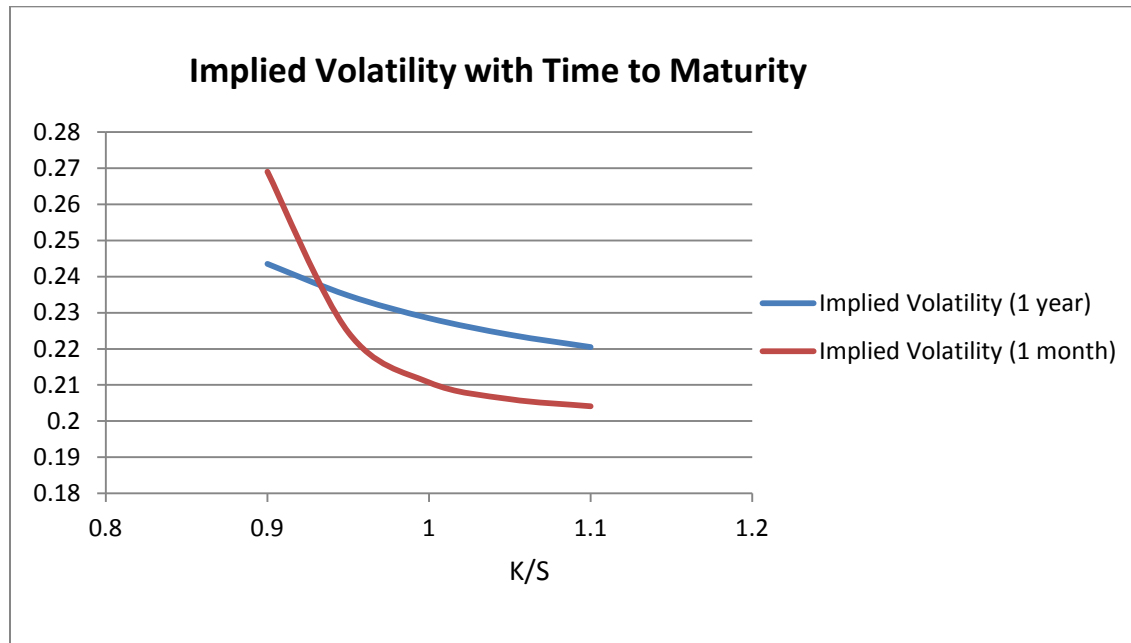


Figure 2

The observed implied volatility skew also has a term-structure. Specifically, the skew tends to be steeper at shorter maturities. To see how the skew changes with analogy making, the previous illustration is repeated with a shorter time to maturity of only 1 month. Figure 3 plots the resulting implied volatility skews both at a longer time to maturity of 1 year and at a considerably shorter maturity of only one month. As can be seen, the skew is steeper at shorter maturity.



**Figure 3**

## **6. The Profitability of Covered Call Writing with Analogy Making**

The profitability of covered call writing is quite puzzling in the Black Scholes framework. Whaley (2002) shows that BXM (a Buy Write Monthly Index tracking a Covered Call on S&P 500) has significantly lower volatility when compared with the index, however, it offers nearly the same return as the index. In the Black Scholes framework, the covered call strategy is expected to have lower risk as well as lower return when compared with buying the index only. See Black (1975). In fact, in an efficient market, the risk adjusted return from covered call writing should be no different than the risk adjusted return from just holding the index.

The covered call strategy (S denotes stock, C denotes call) is given by:

$$V = S - C$$

With analogy making, this is equal to:

$$\begin{aligned} V &= S - e^{-|\epsilon_K|(1+\theta)(T-t)} \{SN(d_1^A) - Ke^{-(r+\delta)(T-t)}N(d_2^A)\} \\ \Rightarrow V &= \left(1 - e^{-|\epsilon_K|(1+\theta)(T-t)}N(d_1^A)\right)S + e^{-|\epsilon_K|(1+\theta)(T-t)}N(d_2^A)Ke^{-(r+\delta)(T-t)} \end{aligned} \quad (6.1)$$

The corresponding value under the Black Scholes assumptions is:

$$V = (1 - N(d_1))S + N(d_2)Ke^{-r(T-t)} \quad (6.2)$$

A comparison of 6.1 and 6.2 shows that covered call strategy is expected to perform much better with analogy making when compared with its expected performance in the Black Scholes world. With analogy making, covered call strategy creates a portfolio which is equivalent to having a portfolio with a weight of  $1 - e^{-|\epsilon_K|(1+\theta)(T-t)}N(d_1^A)$  on the stock and a weight of  $N(d_2^A)$  on a hypothetical risk free asset with a return of  $r + \delta + |\epsilon_K|(1 + \theta)$ . The stock has a return of  $r + \delta$  plus dividend yield. This implies that, with analogy making, the return from covered call strategy is expected to be comparable to the return from holding the underlying stock only. The presence of a hypothetical risk free asset in 6.1 implies that the standard deviation of covered call returns is lower than the standard deviation from just holding the underlying stock. Hence, the superior historical performance of covered call strategy is consistent with analogy making.

### ***6.1 The Zero-Beta Straddle Performance with Analogy Making***

Another empirical puzzle in the Black-Scholes/CAPM framework is that zero beta straddles lose money. Goltz and Lai (2009), Coval and Shumway (2001) and others find that zero beta straddles earn negative returns on average. This is in sharp contrast with the Black-Scholes/CAPM prediction which says that the zero-beta straddles should earn the risk free rate. A zero-beta straddle is constructed by taking a long position in corresponding call and put options with weights chosen so as to make the portfolio beta equal to zero:



$$\theta \cdot \beta_{Call} + (1 - \theta) \cdot \beta_{Put} = 0$$

$$\Rightarrow \theta = \frac{-\beta_{Put}}{\beta_{Call} - \beta_{Put}}$$

$$\text{Where } \beta_{Call} = N(d_1) \cdot \frac{Stock}{Call} \cdot \beta_{Stock} \text{ and } \beta_{Put} = (N(d_1) - 1) \cdot \frac{Stock}{Put} \cdot \beta_{Stock}$$

It is straightforward to show that with analogy making, where call and put prices are determined in accordance with proposition 4, the zero-beta straddle earns a significantly smaller return than the risk free rate (with returns being negative for a wide range of realistic parameter values). See Appendix H for proof. Intuitively, with analogy making, both call and put options are more expensive when compared with Black-Scholes prices. Hence, the returns are smaller.

Analogy based option pricing not only generates the implied volatility skew, it is also consistent with key empirical findings regarding option portfolio returns such as covered call writing and zero-beta straddles.

## 7. Leverage Adjusted Option Returns with Analogy Making

Leverage adjustment dilutes beta risk of an option by combining it with a risk free asset. Leverage adjustment combines each option with a risk-free asset in such a manner that the overall beta risk becomes equal to the beta risk of the underlying stock. The weight of the option in the portfolio is equal to its inverse price elasticity w.r.t the underlying stock's price:

$$\beta_{portfolio} = \Omega^{-1} \times \beta_{call} + (1 - \Omega^{-1}) \times \beta_{riskfree}$$

$$\text{where } \Omega = \frac{\partial Call}{\partial Stock} \times \frac{Stock}{Call} \text{ (i.e price elasticity of call w.r.t the underlying stock)}$$

$$\beta_{call} = \Omega \times \beta_{stock}$$

$$\beta_{riskfree} = 0$$

$$\Rightarrow \beta_{portfolio} = \beta_{stock}$$

Constantinides, Jackwerth and Savov (2013) uncover a number of puzzling empirical facts regarding leverage adjusted index option returns. They find that over a period ranging from April

1986 to January 2012, the average percentage monthly returns of *leverage-adjusted index call and put options are decreasing in the ratio of strike to spot*. They also find that *leverage adjusted put returns are larger than the corresponding leverage adjusted call returns*. The empirical findings in Contantinides et al (2012) are inconsistent with the Black-Scholes/CAPM framework, which predicts that the leverage adjusted returns should be equal to the return from the underlying index. That is, they should not fall with strike, and the leverage adjusted put option returns should not be any different than the leverage adjusted call returns.

If analogy making determines call prices, then the behavior of leverage adjusted call and put returns should be a lot different than their predicted behavior under the Black-Scholes assumptions. For call options (suppressing subscripts for simplicity):

$$\Omega^{-1} \cdot \frac{1}{dt} \left[ \frac{E[dC]}{C} \right] + (1 - \Omega^{-1})r \quad (7.1)$$

$$\text{where } E[dC] = \left( (r + \delta)S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt \quad (7.2)$$

According to the analogy based PDE (with  $\theta = 0$ ):

$$(r + \delta + |\epsilon|)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} (r + \delta)S + \frac{\partial^2 C}{\partial S^2} \frac{\sigma^2 S^2}{2} \quad (7.3)$$

Substituting (7.3) and (7.2) in (7.1) and simplifying leads to:

$$\text{Leverage Adjusted Call Return} = \Omega^{-1}(\delta + |\epsilon|) + r \quad (7.4)$$

Note that in (7.4), if  $|\epsilon| = (\Omega - 1)\delta$ , that is if the adjustment is correct (no anchoring bias), then the leverage adjusted call return is equal to the return from the underlying stock. With the anchoring bias, that is, when  $|\epsilon| < (\Omega - 1)\delta$ , the leverage adjusted call return falls with strike-to-spot. To take an example, suppose the adjustment is only a quarter of the correct adjustment. That is, assume that  $|\epsilon| = 0.25(\Omega - 1)\delta$ . It follows:

$$\text{Leverage Adjusted Call Return} = 0.25 + \frac{0.75\delta}{\Omega} + r \quad (7.4b)$$

As  $\Omega$  rises rapidly with strike-to-spot ratio, leverage adjusted call return falls as strike-to-spot ratio rises. Hence, analogy based option pricing is consistent with empirical evidence regarding leverage adjusted call returns.

The corresponding leverage adjusted put option return with analogy based option pricing is:

$$\text{Leverage Adjusted Put Return} = r + \left\{ \frac{\delta S - (\delta + |\epsilon|)C}{S(1 - \Delta_{call})} \right\} \quad (7.5)$$

Where  $\Delta_{call} = \frac{\partial C}{\partial S}$

If there is no anchoring bias, that is, if the adjustment is correct, then  $|\epsilon| = (\Omega - 1)\delta$ . Substituting this value in (7.5) leads to leverage adjusted put return being equal to the return from the underlying stock. What happens if  $|\epsilon| < (\Omega - 1)\delta$ ? To fix ideas, as before, let's take the following example:

$|\epsilon| = 0.25(\Omega - 1)\delta$ . (7.5) then becomes:

$$\text{Leverage Adjusted Put Return} = r + \delta \left\{ \frac{S - (0.75 + 0.25\Omega)C}{S(1 - \Delta_{call})} \right\} \quad (7.5b)$$

It is straightforward to check that for realistic parameter values, (7.5b) falls with strike-to-spot, and (7.5b) is larger than (7.4b). Hence, empirical findings in Constantinides et al (2012) regarding leverage adjusted option returns are consistent with analogy based option pricing.

## 8. Analogy based Option Pricing with Stochastic Volatility

In this section, we put forward an analogy based option pricing model for the case when the underlying stock price and its instantaneous variance are assumed to obey the uncorrelated stochastic processes described in Hull and White (1987):

$$dS = \mu S dt + \sqrt{V} S dw$$

$$dV = \varphi V dt + \varepsilon V dz$$

$$E[dw dz] = 0$$

Where  $V = \sigma^2$  (Instantaneous variance of stock's returns), and  $\varphi$  and  $\varepsilon$  are non-negative constants.  $dw$  and  $dz$  are standard Gauss-Weiner processes that are uncorrelated. Time subscripts in  $S$  and  $V$

are suppressed for notational simplicity. If  $\varepsilon = 0$ , then the instantaneous variance is a constant, and we are back in the Black-Scholes world. Bigger the value of  $\varepsilon$ , which can be interpreted as the volatility of volatility parameter, larger is the departure from the constant volatility assumption of the Black-Scholes model. For simplicity, we assume that the transaction cost  $\theta$  is zero.

Hull and White (1987) is among the first option pricing models that allowed for stochastic volatility. A variety of stochastic volatility models have been proposed including Stein and Stein (1991), and Heston (1993) among others. Here, we use Hull and White (1987) assumptions to show that the idea of analogy making is easily combined with stochastic volatility. Clearly, with stochastic volatility it does not seem possible to form a hedge portfolio that eliminates risk completely because there is no asset which is perfectly correlated with  $V = \sigma^2$ .

If analogy making determines call prices and the underlying stock and its instantaneous volatility follow the stochastic processes described above, then the European call option price (no dividends on the underlying stock for simplicity) must satisfy the partial differentiation equation given below (see Appendix F for the derivation):

$$\frac{\partial C}{\partial t} + (r + \delta)S \frac{\partial C}{\partial S} + \varphi V \frac{\partial C}{\partial V} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \varepsilon^2 V^2 \frac{\partial^2 C}{\partial V^2} = (r + \delta + |\varepsilon|)C \quad (8.1)$$

Where  $\delta$  is the risk premium that a marginal investor in the call option expects to get from the underlying stock.

By definition, under analogy making, the price of the call option is the expected terminal value of the option discounted at the rate which the marginal investor in the option expects to get from investing in the option. The price of the option is then:

$$C(S_t, \sigma_t^2, t) = e^{-(r+\delta+|\varepsilon|)(T-t)} \int C(S_T, \sigma_T^2, T) p(S_T | S_t, \sigma_t^2) dS_T \quad (8.2)$$

Where the conditional distribution of  $S_T$  as perceived by the marginal investor is such that  $E[S_T | S_t, \sigma_t^2] = S_t e^{(r+\delta)(T-t)}$  and  $C(S_T, \sigma_T^2, T)$  is  $\max(S_T - K, 0)$ .

By defining  $\bar{V} = \frac{1}{T-t} \int_t^T \sigma_\tau^2 d\tau$  as the means variance over the life of the option, the distribution of  $S_T$  can be expressed as:

$$p(S_T|S_t, \sigma_t^2) = \int f(S_T|S_t, \bar{V}) g(\bar{V}|S_t, \sigma_t^2) d\bar{V} \quad (8.3)$$

Substituting (8.3) in (8.2) and re-arranging leads to:

$$C(S_t, \sigma_t^2, t) = \int \left[ e^{-(r+\delta+|\epsilon|)(T-t)} \int C(S_T) f(S_T|S_t, \bar{V}) dS_T \right] g(\bar{V}|S_t, \sigma_t^2) d\bar{V} \quad (8.4)$$

By using an argument that runs in parallel with the corresponding argument in Hull and White (1987), it is straightforward to show that the term inside the square brackets is the analogy making price of the call option with a constant variance  $\bar{V}$ . Denoting this price by  $Call_{AM}(\bar{V})$ , the price of the call option under analogy making when volatility is stochastic (as in Hull and White (1987)) is given by (proof available from author):

$$C(S_t, \sigma_t^2, t) = \int Call_{AM}(\bar{V}) g(\bar{V}|S_t, \sigma_t^2) d\bar{V} \quad (8.5)$$

Where  $Call_{AM}(\bar{V}) = e^{-|\epsilon|(T-t)} \{SN(d_1^M) - Ke^{-(r+\delta)(T-t)} N(d_2^M)\}$

$$d_1^M = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}; d_2^M = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \delta - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Equation (8.5) shows that the analogy based call option price with stochastic volatility is the analogy based price with constant variance integrated with respect to the distribution of mean volatility.

### 8.1 Option Pricing Implications

Stochastic volatility models require a strong correlation between the volatility process and the stock price process in order to generate the implied volatility skew. They can only generate a more symmetric U-shaped smile with zero correlation as assumed here. In contrast, the analogy making stochastic volatility model (equation 8.5) can generate a variety of skews and smiles even with zero correlation. What type of implied volatility structure is ultimately seen depends on the parameters  $\delta$  and  $\epsilon$  assuming that  $|\epsilon|$  is below a certain threshold. It is easy to see that if  $\epsilon = 0$  and  $\delta > 0$ , only the implied volatility skew is generated, and if  $\delta = 0$  and  $\epsilon > 0$ , only a more symmetric smile arises. For positive  $\delta$ , there is a threshold value of  $\epsilon$  below which skew arises and above which smile takes

shape. Typically, for options on individual stocks, the smile is seen, and for index options, the skew arises. The approach developed here provides a potential explanation for this as  $\varepsilon$  is likely to be lower for indices due to inbuilt diversification (giving rise to skew) when compared with individual stocks.

## 9. Analogy based Option Pricing with Jump Diffusion

In this section, we integrate the idea of analogy making with the jump diffusion model of Merton (1976). As before, the point is that the idea of analogy making is independent of the distributional assumptions that are made regarding the behavior of the underlying stock. In the previous section, analogy making is combined with the Hull and White stochastic volatility model to illustrate the same point.

Merton (1976) assumes that the stock returns are a mixture of geometric Brownian motion and Poisson-driven jumps:

$$dS = (\mu - \gamma\beta)Sdt + \sigma Sdz + dq$$

Where  $dz$  is a standard Guass-Weiner process, and  $q(t)$  is a Poisson process.  $dz$  and  $dq$  are assumed to be independent.  $\gamma$  is the mean number of jump arrivals per unit time,  $\beta = E[Y - 1]$  where  $Y - 1$  is the random percentage change in the stock price if the Poisson event occurs, and  $E$  is the expectations operator over the random variable  $Y$ . If  $\gamma = 0$  (hence,  $dq = 0$ ) then the stock price dynamics are identical to those assumed in the Black Scholes model. For simplicity, assume that  $E[Y] = 1$ .

The stock price dynamics then become:

$$dS = \mu Sdt + \sigma Sdz + dq$$

Clearly, with jump diffusion, the Black-Scholes no-arbitrage technique cannot be employed as there is no portfolio of stock and options which is risk-free. However, with analogy making, the price of the option can be determined as the return on the call option demanded by the marginal investor is equal to the return he expects from the underlying stock plus an adjustment term.

If analogy making determines the price of the call option when the underlying stock price dynamics are a mixture of a geometric Brownian motion and a Poisson process as described earlier, then the following partial differential equation must be satisfied (see Appendix G for the derivation):

$$\frac{\partial C}{\partial t} + (r + \delta)S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \gamma E[C(SY, t) - C(S, t)] = (r + \delta + |\epsilon|)C \quad (9.1)$$

If the distribution of  $Y$  is assumed to log-normal with a mean of 1 (assumed for simplicity) and a variance of  $v^2$  then by using an argument analogous to Merton (1976), the following analogy based option pricing formula for the case of jump diffusion is easily derived (proof available from author):

$$Call = \sum_{j=0}^{\infty} \frac{e^{-\gamma(T-t)} (\gamma(T-t))^j}{j!} Call_{AM}(S, (T-t), K, r, \delta, \sigma_j) \quad (9.2)$$

$$Call_{AM}(S, (T-t), K, r, \delta, \sigma_j) = e^{-|\epsilon|(T-t)} \{SN(d_1^M) - Ke^{-(r+\delta)(T-t)}N(d_2^M)\}$$

$$d_1^M = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \delta + \frac{\sigma_j^2}{2}\right)(T-t)}{\sigma_j \sqrt{T-t}} \quad d_2^M = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \delta - \frac{\sigma_j^2}{2}\right)(T-t)}{\sigma_j \sqrt{T-t}}$$

$$\sigma_j = \sqrt{\sigma^2 + v^2 \left(\frac{j}{T-t}\right)} \text{ and } v^2 = \frac{f\sigma^2}{\gamma}$$

Where  $f$  is the fraction of volatility explained by jumps.

The formula in (9.2) is identical to the Merton jump diffusion formula except for one parameter,  $\delta$ , which is the risk premium that a marginal investor in the call option expects from the underlying stock.

### 9.1 Option Pricing Implications

Merton's jump diffusion model with symmetric jumps around the current stock price can only produce a symmetric smile. Generating the implied volatility skew requires asymmetric jumps (jump mean becomes negative) in the model. However, with analogy making, both the skew and the smile

can be generated even when jumps are symmetric. In particular, for low values of  $\delta$ , a more symmetric smile is generated, and for larger values of  $\delta$ , skew arises.

Even if we one assumes an asymmetric jump distribution around the current stock price, Merton formula, when calibrated with historical data, generates a skew which is a lot less pronounced (steep) than what is empirically observed. See Andersen and Andreasen (2002). The skew generated by the analogy formula (with asymmetric jumps) is typically more pronounced (steep) when compared with the skew without analogy making. Hence, analogy making potentially adds value to a jump diffusion model.

## 10. Conclusions

Intriguing option pricing puzzles include: 1) the implied volatility skew, 2) superior historical performance of covered call writing, 3) worse-than-expected performance of zero beta straddles, and 4) puzzling findings regarding leverage adjusted index option returns. Furthermore, it is well known that average put returns are more negative than what theory predicts, and average call returns are smaller than what one would expect given their systematic risk.

If the risk of the underlying stock is used as a starting point which gets adjusted upwards to arrive at call option risk, then the anchoring bias implies that such adjustments are insufficient. There is considerable field and experimental evidence of the role of anchoring in option pricing. In this article, an anchoring-adjusted option pricing model is put forward. The model provides a unified explanation for the puzzles mentioned above.

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## Appendix A

### Proof of Proposition 1

Start by considering  $|\epsilon| = 0$  first.

For case 1, when  $X_1 > X_2 > K$ , the results follow from a direct comparison of (3.4) and (3.5).

For case 2, when  $X_1 > K > X_2$ , the spectrum of possibilities is further divided into three sub-classes and the results are proved for each sub-class one by one. The three sub-classes are: (i)  $K = \frac{X_1 + X_2}{2}$ , (ii)  $X_2 < K < \frac{X_1 + X_2}{2}$ , and (iii)  $X_1 > K > \frac{X_1 + X_2}{2}$ .

**Case 2 sub-class (i):  $K = \frac{X_1 + X_2}{2}$**

If we assume that  $S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \leq \frac{X_1 - K}{X_1 - X_2} (S - X_2)$ , we arrive at a contradiction as follows:

Substitute  $S = f \cdot \frac{X_1+X_2}{2}$  and  $K = \frac{X_1+X_2}{2}$  above and simplify, it follows that  $f \geq 1$ , which is a contradiction as  $f < 1$  if the risk premium is positive.

**Case 2 sub-class (ii):**  $X_2 < K < \frac{X_1+X_2}{2}$  or equivalently  $K = g \frac{X_1+X_2}{2}$  where  $\frac{2X_2}{X_1+X_2} < g < 1$

If we assume that  $S \cdot \frac{X_1}{X_1+X_2} - \frac{K}{2} \cdot f \leq \frac{X_1-K}{X_1-X_2} (S - X_2)$ , we arrive at a contradiction as follows:

Substitute  $S = f \cdot \frac{X_1+X_2}{2}$  and  $K = g \frac{X_1+X_2}{2}$  above and simplify, it follows that  $X_1 \leq X_2$ , which is a contradiction.

**Case 2 sub-class (iii):**  $X_1 > K > \frac{X_1+X_2}{2}$  or equivalently  $K = g \frac{X_1+X_2}{2}$  where  $1 < g < \frac{2X_1}{X_1+X_2}$

Similar logic as used in the case above leads to a contradiction:  $X_1 \leq X_2$ . Hence, the analogy price must be larger than the no-arbitrage price if the risk premium is positive.

If  $|\epsilon| > 0$ , then the threshold values are given below:

$$|\epsilon| < \frac{1}{f} \left\{ \frac{S - Kf}{S - K} - 1 \right\} \quad \text{if } X_1 > X_2 > K$$

$$|\epsilon| < \frac{1}{f} \left\{ \frac{S \frac{X_1}{X_1+X_2} - \frac{K}{2} f}{\frac{X_1-K}{X_1-X_2} (S - X_2)} - 1 \right\} \quad \text{if } X_1 > K > X_2$$

## Appendix B

### Proof of Proposition 2

If  $X_1 > X_2 > K$  then there is no-arbitrage if the following holds:

$$\left\{ \frac{S}{1+c} - \frac{K}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{K}{1-c} \right\} \leq S - Kf \leq \left\{ \frac{S}{1-c} - \frac{K}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{K}{1+c} \right\}$$

Realizing that  $S - Kf \geq S - K > \left\{ \frac{S}{1+c} - \frac{K}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{K}{1-c} \right\}$  if  $\delta \geq 0$  and simplifying

$$S - Kf \leq \left\{ \frac{S}{1-c} - \frac{K}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{K}{1+c} \right\} \text{ leads to inequality (3.12).}$$

If  $X_1 > K > X_2$  then there is no-arbitrage if the following holds:

$$\begin{aligned} \left( \frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1+c} - \frac{X_2}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{X_2}{1-c} \right\} &\leq S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \\ &\leq \left( \frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1-c} - \frac{X_2}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{X_2}{1+c} \right\} \end{aligned}$$

Realizing that

$$\begin{aligned} \left( \frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1+c} - \frac{X_2}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{X_2}{1-c} \right\} &\leq \\ \frac{X_1 - K}{X_1 - X_2} (S - X_2) &\leq S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \text{ if } \delta \geq 0 \end{aligned}$$

And simplifying  $S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \leq \left( \frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1-c} - \frac{X_2}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{X_2}{1+c} \right\}$  leads to (3.13).

It is straightforward to check that if  $|\epsilon| > 0$ , then the right side limits are larger.

## Appendix C

### Proof of Proposition 3

Start by considering the situation when  $|\epsilon| = 0$ .

#### Case 1: $X_1 > X_2 > K$

Delta-hedged portfolio is  $Sx - C$ . In this case,  $x = 1$ ,  $S = f \cdot \frac{X_1 + X_2}{2}$ , and  $C = S - Kf$

If the red state is realized,  $S - C$  changes from  $Kf$  to  $K$ . If the blue state is realized  $S - C$  also changes from  $Kf$  to  $K$ . Hence, the growth rate is equal to  $\frac{1}{f} - 1$  in either state.

#### Case 2: $X_1 > K > X_2$

Delta-hedged portfolio is  $Sx - C$ . In this case,  $x = \frac{X_1 - K}{X_1 - X_2}$ ,  $S = f \cdot \frac{X_1 + X_2}{2}$ , and

$$C = S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f$$

Consider three sub-classes and prove the result for each: (i)  $K = \frac{X_1 + X_2}{2}$ , (ii)  $X_2 < K < \frac{X_1 + X_2}{2}$ , and

(iii)  $X_1 > K > \frac{X_1 + X_2}{2}$ . For the first sub-class the delta-hedged portfolio changes from the initial value

of  $f \frac{X_2}{2}$  to  $\frac{X_2}{2}$  in both the red and the blue states. Hence, the growth rate is equal to  $\frac{1}{f} - 1$  in either

state. For the second and third sub-classes, the delta-hedged portfolio changes from

$$\frac{f\{(2-g)X_1X_2-gX_2^2\}}{2(X_1-X_2)} \text{ to } \frac{\{(2-g)X_1X_2-gX_2^2\}}{2(X_1-X_2)} \text{ in both red and blue states. Hence, the growth rate is equal to}$$

$$\frac{1}{f} - 1.$$

By using similar logic, it is straightforward to see that the growth rate falls as  $|\epsilon|$  rises, with the growth rate becoming equal to  $r$  if  $|\epsilon| = (\Omega - 1)\delta$ .

## Appendix D

The analogy based PDE derived in Appendix D can be solved by converting to heat equation and exploiting its solution. The steps are identical to the derivation of the Black Scholes model with the risk free rate  $r$ , replaced with  $r + \delta$ , and an additional term due to  $\epsilon$ .

Start by making the following transformations in (4.4):

$$\tau = \frac{\sigma^2}{2}(T - t)$$

$$x = \ln \frac{S}{K} \Rightarrow S = Ke^x$$

$$C(S, t) = K \cdot c(x, \tau) = K \cdot c\left(\ln\left(\frac{S}{K}\right), \frac{\sigma^2}{2}(T - t)\right)$$

It follows,

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \left(-\frac{\sigma^2}{2}\right)$$

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 C}{\partial S^2} = K \cdot \frac{1}{S^2} \cdot \frac{\partial^2 c}{\partial x^2} - K \cdot \frac{1}{S^2} \frac{\partial C}{\partial x}$$

Plugging the above transformations into (4.4) and writing  $\tilde{r} = \frac{2(r+\delta)}{\sigma^2}$ , and  $\tilde{\epsilon} = \frac{2|\epsilon|(1+\theta)}{\sigma^2}$  we get:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\tilde{r} - 1) \frac{\partial c}{\partial x} - (\tilde{r} + \tilde{\epsilon})c \quad (D1)$$

With the boundary condition/initial condition:

$$C(S, T) = \max\{S - K, 0\} \text{ becomes } c(x, 0) = \max\{e^x - 1, 0\}$$

To eliminate the last two terms in (D1), an additional transformation is made:

$$c(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau)$$

It follows,

$$\begin{aligned} \frac{\partial c}{\partial x} &= \alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x} \\ \frac{\partial^2 c}{\partial x^2} &= \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial c}{\partial \tau} &= \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial \tau} \end{aligned}$$

Substituting the above transformations in (D1), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\alpha^2 + \alpha(\tilde{r} - 1) - (\tilde{r} + \tilde{\epsilon}) - \beta)u + (2\alpha + (\tilde{r} - 1)) \frac{\partial u}{\partial x} \quad (D2)$$

Choose  $\alpha = -\frac{(\tilde{r}-1)}{2}$  and  $\beta = -\frac{(\tilde{r}+1)^2}{4} - (\tilde{\epsilon})$ . (D2) simplifies to the Heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad (D3)$$

With the initial condition:

$$u(x_0, 0) = \max\{(e^{(1-\alpha)x_0} - e^{-\alpha x_0}), 0\} = \max\left\{\left(e^{\left(\frac{\tilde{r}+1}{2}\right)x_0} - e^{\left(\frac{\tilde{r}-1}{2}\right)x_0}\right), 0\right\}$$

The solution to the Heat equation in our case is:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4\tau}} u(x_0, 0) dx_0$$

Change variables:  $= \frac{x_0 - x}{\sqrt{2\tau}}$ , which means:  $dz = \frac{dx_0}{\sqrt{2\tau}}$ . Also, from the boundary condition, we know that  $u > 0$  iff  $x_0 > 0$ . Hence, we can restrict the integration range to  $z > \frac{-x}{\sqrt{2\tau}}$

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}+1}{2}\right)(x+z\sqrt{2\tau})} dz - \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}-1}{2}\right)(x+z\sqrt{2\tau})} dz$$

$$=: H_1 - H_2$$

Complete the squares for the exponent in  $H_1$ :

$$\begin{aligned} \frac{\tilde{r}+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2} &= -\frac{1}{2}\left(z - \frac{\sqrt{2\tau}(\tilde{r}+1)}{2}\right)^2 + \frac{\tilde{r}+1}{2}x + \tau \frac{(\tilde{r}+1)^2}{4} \\ &=: -\frac{1}{2}y^2 + c \end{aligned}$$

We can see that  $dy = dz$  and  $c$  does not depend on  $z$ . Hence, we can write:

$$H_1 = \frac{e^c}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau} - \sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

A normally distributed random variable has the following cumulative distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{y^2}{2}} dy$$

$$\text{Hence, } H_1 = e^c N(d_1) \text{ where } d_1 = x/\sqrt{2\tau} + \sqrt{\tau/2}(\tilde{r}+1)$$

$$\text{Similarly, } H_2 = e^f N(d_2) \text{ where } d_2 = x/\sqrt{2\tau} + \sqrt{\tau/2}(\tilde{r}-1) \text{ and } f = \frac{\tilde{r}-1}{2}x + \tau \frac{(\tilde{r}-1)^2}{4}$$

The analogy based European call pricing formula is obtained by recovering original variables:

$$C = e^{-|\epsilon|(1+\theta)(T-t)} \{SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)\}$$



Where  $d_1 = \frac{\ln(S/K) + (r + \delta + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$  and  $d_2 = \frac{\ln(\frac{S}{K}) + (r + \delta - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$

### Appendix F

By Ito's Lemma (time subscript is suppressed for simplicity):

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial V} dV + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} dt + \frac{1}{2} V^2 \varepsilon^2 \frac{\partial^2 C}{\partial V^2} dt \quad (F1)$$

Substituting:

$$dS = \mu S dt + \sigma S dw \text{ and,}$$

$$dV = \varphi V dt + \varepsilon V dz$$

in (F1) and taking expectations leads to:

$$\frac{\partial C}{\partial t} + (r + \delta) S \frac{\partial C}{\partial S} + \varphi V \frac{\partial C}{\partial V} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \varepsilon^2 V^2 \frac{\partial^2 C}{\partial V^2} = (r + \delta + |\varepsilon|) C \quad (F2)$$

### Appendix G

By following a very similar argument as in appendix F, and using Ito's lemma for the continuous part and an analogous Lemma for the discontinuous part, the following is obtained:

$$\frac{\partial C}{\partial t} + (r + \delta) S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \gamma E[C(SY, t) - C(S, t)] = (r + \delta + |\varepsilon|) C$$

### Appendix H

For simplicity and without loss of generality, assume that the transaction cost  $\theta$  is 0. Following Coval and Shumway (2001) and some algebraic manipulations, the return from a zero-beta-straddle can be written as:

$$r_{straddle} = \frac{-\Omega_c C + S}{\Omega_c P - \Omega_c C + S} \cdot r_{call} + \frac{\Omega_c P + S}{\Omega_c P - \Omega_c C + S} \cdot r_{put}$$

Where  $C$  and  $P$  denote call and put prices respectively,  $r_{call}$  is expected call return and  $r_{put}$  is expected put return.

Under analogy making:

$$r_{call} = \mu + |\varepsilon|$$

$$r_{put} = \frac{(\mu + |\epsilon|)C - \mu S + rPV(K)}{P}$$

Substituting  $r_{call}$  and  $r_{put}$  in the expression for  $r_{straddle}$ , and simplifying implies that as long as the risk premium on the underlying is positive and  $|\epsilon|$  is not very large (not large enough to reach the correct risk judgment), it follows that:

$$r_{straddle} < r$$

## Appendix I

Note that for a put option, if the underlying stock has a positive risk premium, then the expected put payoff must be less than its price. That is, expected put return is negative. The proof follows directly from realizing that if the risk premium on the underlying stock is positive and  $|\epsilon_K|(1 + \theta)$  is below a certain threshold, the price of a put option under analogy making is larger than the price of a put option in the Black Scholes model. The threshold value is:

$$|\epsilon|(T - t) \leq \left\{ \frac{SN(d_1^A) - Ke^{-(r+\delta)(T-t)}N(d_2^A)}{SN(d_1^{BS}) - Ke^{-r(T-t)}N(d_2^{BS})} \right\} \cdot \frac{1}{1 + \theta}$$

$d_1^A = \frac{\ln(S/K) + (r + \delta + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ ,  $d_2^A = \frac{\ln(S/K) + (r + \delta - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ ,  $d_1^{BS}$  and  $d_2^{BS}$  are corresponding Black-Scholes arguments.